SISTEM ANTRIAN *M*/*M*/*c*/GD/K/∞

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Sistem antrian dengan waktu antardatang berdistribusi eksponensial atau jumlah pelanggan yang datang berdistribusi Poisson. Waktu layannya berdistribusi eksponensial atau jumlah pelanggan yang berangkat berdistribusi Poisson. Jumlah pelayan paralelnya sebanyak *c*. Disiplin pelayanannya umum. Jumlah pelanggan maksimum yang diperbolehkan dalam sistem sebanyak *K*. Jumlah populasi pelanggan takhingga. Sistem antrian ini dinotasikan dengan sistem antrian $M/M/c/\mathrm{GD}/K/\infty$, di mana
$K \geq c$

Laju datang

$$\lambda_n = \begin{cases} \lambda & \text{if } 0 \leq n < K \\ 0 & \text{if } n \geq K \\ \text{"Tidak didefinisikan"} & \text{otherwise} \end{cases}$$

di mana *lamda* menyatakan laju datang (*arrival rate*) yaitu jumlah pelanggan yang datang rata-rata per satuan waktu.

Lamda n menyatakan laju datang keadaan jumlah pelanggan sebanyak *n*.

Laju layan

$$\mu_n = \begin{cases} n\,\mu & \text{if } 0 \leq n < c \\ c\,\mu & \text{if } c \leq n \leq K \\ 0 & \text{if } n > K \\ \text{"Tidak didefinisikan"} & \text{otherwise} \end{cases}$$

Mu menyatakan laju layan yaitu jumlah pelanggan yang dilayani rata-rata per satuan waktu.

Mu n menyatakan laju layan keadaan jumlah pelanggan sebanyak *n*.

Laju datang rata-rata effektif

$$\lambda_{eff} = \sum_{n=0}^{K} \left(\lambda_n\, p_n\right)$$

$$\lambda_{eff} = \lambda_0\, p_0 + \lambda_1\, p_1 + \ldots + \lambda_{K-1}\, p_{K-1} + \lambda_K\, p_K$$

sedangkan

$$\lambda_n = \begin{cases} \lambda & \text{if } 0 \leq n < K \\ 0 & \text{if } n \geq K \\ \text{"Tidak didefinisikan"} & \text{otherwise} \end{cases}$$

sehingga

$$\lambda_{eff} = \lambda\ p_0 + \lambda\ p_1 + ... + \lambda\ p_{K-1} + 0\ p_K$$

$$\lambda_{eff} = \lambda\ \left(p_0 + p_1 + ... + p_{K-1}\right)$$

karena

$$\sum_{n=0}^{K} p_n = 1 \qquad \text{atau} \qquad \sum_{n=0}^{K-1} p_n = 1 - p_K$$

sehingga

$$\lambda_{eff} = \lambda\ \left(1 - p_K\right)$$

Di mana p_n menyatakan probabilitas ada n pelanggan sistem antrian dalam keadaan mapan (*steady state*), menyatakan juga ekspektasi proporsi waktu bahwa sistem berada dengan jumlah pelanggan sebanyak n.

Keadaan mapan (*steady state*) berarti distribusi probabilitas jumlah pelanggan dalam antrian dan distribusi probabilitas jumlah pelanggan dalam sistem tidak bergantung waktu.

Jika $0 \le n < c$ di mana $K \ge c$

$$p_n = \frac{\lambda_{n-1}\ \lambda_{n-2}\ \cdots\ \lambda_0}{\mu_n\ \mu_{n-1}\ \cdots\ \mu_1}\ p_0$$

sedangkan

$$\lambda_n = \begin{vmatrix} \lambda \quad if\ \ 0 \le n < K \\ 0 \quad if\ \ n \ge K \\ \text{"Tidak didefinisikan"} \quad otherwise \end{vmatrix}$$

dan

$$\mu_n = \begin{vmatrix} n\ \mu \quad if\ \ 0 \le n < c \\ c\ \mu \quad if\ \ c \le n \le K \\ 0 \quad if\ \ n > K \\ \text{"Tidak didefinisikan"} \quad otherwise \end{vmatrix}$$

$$p_n = \frac{\lambda\ \ \lambda\ \ ...\ \lambda}{(n\ \mu)\ \ [(n-1)\ \mu]\ \ ...\ \mu}\ p_0$$

$$p_n = \frac{\lambda^n}{[n\ (n-1)\ ...\ 1]\ \ (\mu\ \ \mu\ \ ...\ \mu)}\ p_0$$

$$p_n = \frac{\lambda^n}{n!\ \mu^n}\ p_0$$

$$p_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\ p_0$$

Jika $c \le n \le K$ di mana $K \ge c$

$$p_n = \frac{\lambda_{n-1}\ \lambda_{n-2}\ \cdots\ \lambda_0}{\mu_n\ \mu_{n-1}\ \cdots\ \mu_1}\ p_0$$

sedangkan

$$\lambda_n = \begin{vmatrix} \lambda \quad \text{if}\ \ 0 \le n < K \\ 0 \quad \text{if}\ \ n \ge K \\ \text{"Tidak didefinisikan"} \quad \text{otherwise} \end{vmatrix}$$

dan

$$\mu_n = \begin{vmatrix} n\ \mu \quad \text{if}\ \ 0 \le n < c \\ c\ \mu \quad \text{if}\ \ c \le n \le K \\ 0 \quad \text{if}\ \ n > K \\ \text{"Tidak didefinisikan"} \quad \text{otherwise} \end{vmatrix}$$

maka

$$p_n = \frac{\lambda\ \ \lambda\ \ \ldots\ \lambda}{\mu_n\ \mu_{n-1}\ \cdots\ \mu_{c+1}\ \mu_c\ \mu_{c-1}\ \cdots\ \mu_2\ \mu_1}\ p_0$$

$$p_n = \frac{\lambda^n}{(c\ \mu)\ (c\ \mu)\ \ldots\ (c\ \mu)\ (c\ \mu)\ [(c-1)\ \mu]\ \ldots\ (2\ \mu)\ (\mu)}\ p_0$$

$$p_n = \frac{\lambda^n}{(c\ \mu)\ (c\ \mu)\ \ldots\ (c\ \mu)\ (c\ \mu)\ [(c-1)\ \mu]\ \ldots\ (2\ \mu)\ (\mu)}\ p_0$$

$$p_n = \frac{\lambda^n}{(c\ \mu)^{n-c}\ (c\ \mu)\ [(c-1)\ \mu]\ \ldots\ (2\ \mu)\ (\mu)}\ p_0$$

$$p_n = \frac{\lambda^n}{(c\ \mu)^{n-c}\left[[c\ (c-1)\ \ldots\ (2)\ \ 1]\ \mu^c\right]}\ p_0$$

$$p_n = \frac{\lambda^n}{(c\,\mu)^{n-c}\left(c!\,\mu^c\right)}\,p_0$$

$$p_n = \frac{\lambda^n}{\left(c^{n-c}\,\mu^{n-c}\right)\left(c!\,\mu^c\right)}\,p_0$$

$$p_n = \frac{\lambda^n}{c!\,c^{n-c}\,\mu^n}\,p_0$$

$$p_n = \frac{1}{c!\,c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n p_0$$

Jadi

$$p_n = \begin{cases} \dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n p_0 & \text{if } 0 \le n < c \\ \dfrac{1}{c!\,c^{n-c}}\left(\dfrac{\lambda}{\mu}\right)^n p_0 & \text{if } c \le n \le K \\ 0 & \text{if } n > K \end{cases}$$

Untuk mencari p_0 didapat dari

$$\sum_{n=0}^{K} p_n = 1$$

$$\sum_{n=0}^{c-1}\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n p_0\right] + \sum_{n=c}^{K}\left[\frac{1}{c!\,c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n p_0\right] = 1$$

$$p_0\left[\sum_{n=0}^{c-1}\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \sum_{n=c}^{K}\left[\frac{1}{c!\,c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n\right]\right] = 1$$

$$p_0\left[\sum_{n=0}^{c-1}\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \sum_{n-c=c-c}^{K-c}\left[\frac{1}{c!\,c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n\right]\right] = 1$$

Bila ditulis $n - c = m$ atau $n = m + c$ maka

$$p_0 \left[\sum_{n=0}^{c-1} \left[\frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n \right] + \sum_{m=0}^{K-c} \left[\frac{1}{c!\ c^m} \left(\frac{\lambda}{\mu} \right)^{m+c} \right] \right] = 1$$

$$p_0 \left[\sum_{n=0}^{c-1} \left[\frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n \right] + \frac{1}{c!} \sum_{m=0}^{K-c} \left[\frac{1}{c^m} \left(\frac{\lambda^m \lambda^c}{\mu^m \mu^c} \right) \right] \right] = 1$$

$$p_0 \left[\sum_{n=0}^{c-1} \left[\frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n \right] + \frac{1}{c!} \sum_{m=0}^{K-c} \left[\frac{\lambda^m}{c^m \mu^m} \left(\frac{\lambda^c}{\mu^c} \right) \right] \right] = 1$$

$$p_0 \left[\sum_{n=0}^{c-1} \left[\frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n \right] + \frac{1}{c!} \sum_{m=0}^{K-c} \left[\left(\frac{\lambda}{c\ \mu} \right)^m \left(\frac{\lambda}{\mu} \right)^c \right] \right] = 1$$

$$p_0 \left[\sum_{n=0}^{c-1} \left[\frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n \right] + \frac{1}{c!} \left(\frac{\lambda}{\mu} \right)^c \sum_{m=0}^{K-c} \left[\left(\frac{\lambda}{c\ \mu} \right)^m \right] \right] = 1$$

padahal $$\sum_{m=0}^{K-c} \left[\left(\frac{\lambda}{c\ \mu} \right)^m \right] = \frac{1 - \left(\frac{\lambda}{c\ \mu} \right)^{K-c+1}}{1 - \frac{\lambda}{c\ \mu}}$$ untuk $\frac{\lambda}{c\ \mu} \neq 1$

maka untuk $\frac{\lambda}{c\ \mu} \neq 1$

$$p_0 \left[\sum_{n=0}^{c-1} \left[\frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n \right] + \frac{1}{c!} \left(\frac{\lambda}{\mu} \right)^c \left[\frac{1 - \left(\frac{\lambda}{c\ \mu} \right)^{K-c+1}}{1 - \frac{\lambda}{c\ \mu}} \right] \right] = 1$$

jadi untuk $\frac{\lambda}{c\,\mu} \neq 1$

$$p_0 = \frac{1}{\sum_{n=0}^{c-1}\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left[\frac{1-\left(\frac{\lambda}{c\,\mu}\right)^{K-c+1}}{1-\frac{\lambda}{c\,\mu}}\right]}$$

$\rho = \frac{\lambda}{c\,\mu}$ **dikenal sebagai faktor utilisasi / intensitas lalu lintas.**

Untuk $\frac{\lambda}{c\,\mu} = 1$

$$p_0 \left[\sum_{n=0}^{c-1}\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \sum_{m=0}^{K-c}\left[\left(\frac{\lambda}{c\,\mu}\right)^m\right]\right] = 1$$

$$p_0 \left[\sum_{n=0}^{c-1}\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \sum_{m=0}^{K-c}\left[(1)^m\right]\right] = 1$$

$$p_0 \left[\sum_{n=0}^{c-1}\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c [1 + (K-c)\ 1]\right] = 1$$

$$p_0 \left[\sum_{n=0}^{c-1}\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c (1 + K - c)\right] = 1$$

jadi untuk $\frac{\lambda}{c\,\mu} = 1$

$$p_0 = \frac{1}{\sum_{n=0}^{c-1}\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c (1 + K - c)}$$

Jadi p_0

$$p_0 = \begin{cases} \dfrac{1}{\displaystyle\sum_{n=0}^{c-1} \left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left[\frac{1-\left(\frac{\lambda}{c\,\mu}\right)^{K-c+1}}{1-\frac{\lambda}{c\,\mu}}\right]} & if\ \dfrac{\lambda}{c\,\mu} \neq 1 \\[2ex] \dfrac{1}{\displaystyle\sum_{n=0}^{c-1} \left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c (1+K-c)} & if\ \dfrac{\lambda}{c\,\mu} = 1 \end{cases}$$

Sehingga p_n

$$p_n = \begin{cases} \dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n \dfrac{1}{\displaystyle\sum_{n=0}^{c-1} \left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left[\frac{1-\left(\frac{\lambda}{c\,\mu}\right)^{K-c+1}}{1-\frac{\lambda}{c\,\mu}}\right]} & if\ (0 \le n < c) \wedge \left(\dfrac{\lambda}{c\,\mu} \neq 1\right) \\[2ex] \dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n \dfrac{1}{\displaystyle\sum_{n=0}^{c-1} \left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c (1+K-c)} & if\ (0 \le n < c) \wedge \left(\dfrac{\lambda}{c\,\mu} = 1\right) \\[2ex] \dfrac{1}{c!\,c^{n-c}}\left(\dfrac{\lambda}{\mu}\right)^n \dfrac{1}{\displaystyle\sum_{n=0}^{c-1} \left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left[\frac{1-\left(\frac{\lambda}{c\,\mu}\right)^{K-c+1}}{1-\frac{\lambda}{c\,\mu}}\right]} & if\ (c \le n \le K) \wedge \left(\dfrac{\lambda}{c\,\mu} \neq \right. \\[2ex] \dfrac{1}{c!\,c^{n-c}}\left(\dfrac{\lambda}{\mu}\right)^n \dfrac{1}{\displaystyle\sum_{n=0}^{c-1} \left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c (1+K-c)} & if\ (c \le n \le K) \wedge \left(\dfrac{\lambda}{c\,\mu} = 1\right) \\[2ex] 0 & if\ n > K \end{cases}$$

Probabilitas ada *n* pelanggan dalam sistem antrian *M*/*M*/*c*/GD/*K*/∞ yang keadaannya mapan

p_n adalah probabilitas yang merupakan fungsi dari (n, λ, μ, c, K) sehingga p_n untuk sistem antrian *M*/*M*/*c*/GD/*K*/∞ dapat dinotasikan secara lengkap sebagai:

$$
p(n,\lambda,\mu,c,K) := \left|
\begin{array}{l}
\left(c \leftarrow \dfrac{c}{pelayan}\right) \wedge \left(n \leftarrow \dfrac{n}{pelanggan}\right) \wedge \left(K \leftarrow \dfrac{K}{pelanggan}\right) \\
\text{if } K \ge c \\
\quad \left|
\begin{array}{l}
\text{if } \left(\dfrac{\lambda}{c\,\mu} \ne 1\right) \wedge (n \le K) \\
\quad \left|
\begin{array}{l}
p_0 \leftarrow \dfrac{1}{\left[\displaystyle\sum_{n=0}^{c-1}\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]\right] + \dfrac{1}{c!}\left(\dfrac{\lambda}{\mu}\right)^c \dfrac{1-\left(\dfrac{\lambda}{c\,\mu}\right)^{K-c+1}}{1-\dfrac{\lambda}{c\,\mu}}} \\
\dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n p_0 \quad \text{if } 0 \le n < c \\
\dfrac{1}{c!\,c^{n-c}}\left(\dfrac{\lambda}{\mu}\right)^n p_0 \quad \text{otherwise}
\end{array}\right. \\
\text{if } \left(\dfrac{\lambda}{c\,\mu} = 1\right) \wedge (n \le K) \\
\quad \left|
\begin{array}{l}
p_0 \leftarrow \dfrac{1}{\left[\displaystyle\sum_{n=0}^{c-1}\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]\right] + \dfrac{1}{c!}\left(\dfrac{\lambda}{\mu}\right)^c (1+K-c)} \\
\dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n p_0 \quad \text{if } 0 \le n < c \\
\dfrac{1}{c!\,c^{n-c}}\left(\dfrac{\lambda}{\mu}\right)^n p_0 \quad \text{otherwise}
\end{array}\right. \\
0 \quad \text{otherwise}
\end{array}\right. \\
\text{"K harus lebih besar atau sama dengan c"} \quad \text{otherwise}
\end{array}\right.
$$

Contoh 1

$\lambda := 14\,\frac{pelanggan}{jam}$

laju datang pelanggan per satuan waktu.

λ menyatakan laju datang (*arrival rate*) yaitu jumlah pelanggan yang datang rata-rata per satuan waktu.

Laju datang rata-rata effektif

$$\lambda_{eff} = \sum_{n=0}^{K} \left(\lambda_n\, p_n\right)$$

$\mu := 4\,\frac{pelanggan}{jam}$

laju layan pelanggan per satuan waktu.

μ menyatakan laju layan yaitu jumlah pelanggan yang dilayani rata-rata per satuan waktu.

$c := 2 pelayan$

Jumlah pelanggan maksimum yang diperbolehkan dalam sistem sebanyak K

$K := 7 pelanggan$

$\lambda = 14\,\frac{pelanggan}{jam}$ $\qquad$ $\mu = 4\,\frac{pelanggan}{jam}$

Bila $n := 0 pelanggan$ $\; p(n,\lambda,\mu,c,K) = 4.331 \times 10^{-3}$ angka ini juga menyatakan ekspektasi proporsi waktu bahwa sistem berada dengan jumlah pelanggan $n = 0$.

Bila $n := 7 pelanggan$ $\; p(n,\lambda,\mu,c,K) = 0.435$ angka ini juga menyatakan ekspektasi proporsi waktu bahwa sistem berada dengan jumlah pelanggan $n = 7\,pelanggan$.

$p(0 pelanggan, \lambda, \mu, c, K) = 4.331 \times 10^{-3}$

$p(1 pelanggan, \lambda, \mu, c, K) = 0.015$

$p(2 pelanggan, \lambda, \mu, c, K) = 0.027$

$p(3 pelanggan, \lambda, \mu, c, K) = 0.046$

$p(4 pelanggan, \lambda, \mu, c, K) = 0.081$

$p(5 pelanggan, \lambda, \mu, c, K) = 0.142$

$p(6 pelanggan, \lambda, \mu, c, K) = 0.249$

$p(7 pelanggan, \lambda, \mu, c, K) = 0.435$

$p(8 pelanggan, \lambda, \mu, c, K) = 0$

$p(9 pelanggan, \lambda, \mu, c, K) = 0$

$p(10 pelanggan, \lambda, \mu, c, K) = 0$

Faktor utilisasi / intensitas lalu lintas:

$$\rho(\lambda, \mu, c) := \frac{\lambda}{c\,\mu}$$

$$\rho(\lambda, \mu, c) = 1.75\,\frac{1}{pelayan}$$

$$\lambda = 14\,\frac{pelanggan}{jam}$$

$$\mu = 4\,\frac{pelanggan}{jam}$$

$$c = 2\,pelayan$$

Ekspektasi jumlah pelanggan dalam antrian atau ekspektasi jumlah pelanggan antri

$$EN_q = \sum_{n=c}^{K} \left[(n-c)\,p_n\right]$$

$$EN_q = (c-c)\,p_c + \sum_{n=c+1}^{K} \left[(n-c)\,p_n\right]$$

$$EN_q = \sum_{n=c+1}^{K} \left[(n-c)\,p_n\right]$$

$$EN_q = \sum_{n=c+1}^{K} \left[(n-c)\,p_n\right]$$

sedangkan

$$p_n = \left| \begin{array}{l} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n p_0 \quad \text{if } 0 \le n < c \\ \frac{1}{c!\,c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n p_0 \quad \text{if } c \le n \le K \\ 0 \quad \text{if } n > K \end{array} \right.$$

$$EN_q = \sum_{n=c+1}^{K} \left[(n-c)\left[\frac{1}{c!\,c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n p_0\right]\right]$$

$$EN_q = \sum_{n-c=(c+1)-c}^{K-c} \left[(n-c) \left[\frac{1}{c!\; c^{n-c}} \left(\frac{\lambda}{\mu} \right)^n p_0 \right] \right]$$

bila $n - c = j$ sehingga $n = j + c$ maka

$$EN_q = \sum_{j=1}^{K-c} \left[j \left[\frac{1}{c!\; c^{(j+c)-c}} \left(\frac{\lambda}{\mu} \right)^{j+c} p_0 \right] \right]$$

$$EN_q = \sum_{j=1}^{K-c} \left[j \left[\frac{1}{c!\; c^{j}} \left(\frac{\lambda}{\mu} \right)^{j+c} p_0 \right] \right]$$

$$EN_q = \sum_{j=1}^{K-c} \left[j \left[\frac{1}{c!\; c^{j}} \left(\frac{\lambda}{\mu} \right)^{j+c} p_0 \right] \right]$$

$$EN_q = \sum_{j=1}^{K-c} \left[j \left[\frac{1}{c!\; c^{j}} \left(\frac{\lambda}{\mu} \right)^{j} \left(\frac{\lambda}{\mu} \right)^{c} p_0 \right] \right]$$

$$EN_q = \frac{1}{c!} \left(\frac{\lambda}{\mu} \right)^{c} p_0 \sum_{j=1}^{K-c} \left[j \; \frac{1}{c^j} \left(\frac{\lambda}{\mu} \right)^{j} \right]$$

$$EN_q = \frac{1}{c!} \left(\frac{\lambda}{\mu} \right)^{c} p_0 \sum_{j=1}^{K-c} \left[j \left(\frac{\lambda}{c\,\mu} \right)^{j} \right]$$

Untuk $\frac{\lambda}{c\,\mu} \neq 1$

$$EN_q = \frac{1}{c!} \left(\frac{\lambda}{\mu} \right)^{c} p_0 \sum_{j=1}^{K-c} \left[j \left(\frac{\lambda}{c\,\mu} \right)^{j} \right]$$

$$EN_q = p_0 \frac{1}{c!} \left(\frac{\lambda}{\mu} \right)^{c} \sum_{j=1}^{K-c} \left[j \left(\frac{\lambda}{c\,\mu} \right)^{j-1} \left(\frac{\lambda}{c\,\mu} \right) \right]$$

$$EN_q = p_0 \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{\lambda}{c\,\mu}\sum_{j=1}^{K-c}\left[j\left(\frac{\lambda}{c\,\mu}\right)^{j-1}\right]$$

$$EN_q = p_0 \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{\lambda}{c\,\mu}\sum_{j=1}^{K-c}\left[\frac{d}{d\left(\frac{\lambda}{c\,\mu}\right)}\left[\left(\frac{\lambda}{c\,\mu}\right)^j\right]\right]$$

$$EN_q = p_0 \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{\lambda}{c\,\mu}\left[\frac{d}{d\left(\frac{\lambda}{c\,\mu}\right)}\sum_{j=1}^{K-c}\left(\frac{\lambda}{c\,\mu}\right)^j\right]$$

$$EN_q = p_0 \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{\lambda}{c\,\mu}\left[\frac{d}{d\left(\frac{\lambda}{c\,\mu}\right)}\left[\left(\frac{\lambda}{c\,\mu}\right)^1+\left(\frac{\lambda}{c\,\mu}\right)^2+\ldots+\left(\frac{\lambda}{c\,\mu}\right)^{K-c}\right]\right]$$

$$EN_q = p_0 \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{\lambda}{c\,\mu}\left[\frac{d}{d\left(\frac{\lambda}{c\,\mu}\right)}\left[\frac{\frac{\lambda}{c\,\mu}-\left(\frac{\lambda}{c\,\mu}\right)^{K-c+1}}{1-\frac{\lambda}{c\,\mu}}\right]\right]$$

Perhatikan

$$\frac{d}{d\left(\frac{\lambda}{c\,\mu}\right)}\left[\frac{\frac{\lambda}{c\,\mu}-\left(\frac{\lambda}{c\,\mu}\right)^{K-c+1}}{1-\frac{\lambda}{c\,\mu}}\right]$$

sama dengan

$$\frac{\left[\frac{d}{d\left(\frac{\lambda}{c\,\mu}\right)}\left[\frac{\lambda}{c\,\mu}-\left(\frac{\lambda}{c\,\mu}\right)^{K-c+1}\right]\right]\left(1-\frac{\lambda}{c\,\mu}\right)-\left[\frac{d}{d\left(\frac{\lambda}{c\,\mu}\right)}\left(1-\frac{\lambda}{c\,\mu}\right)\right]\left[\frac{\lambda}{c\,\mu}-\left(\frac{\lambda}{c\,\mu}\right)^{K-c+1}\right]}{\left(1-\frac{\lambda}{c\,\mu}\right)^2}$$

sama dengan

$$\frac{\left[1-(K-c+1)\left(\frac{\lambda}{c\,\mu}\right)^{K-c}\right]\left(1-\frac{\lambda}{c\,\mu}\right)-(-1)\left[\frac{\lambda}{c\,\mu}-\left(\frac{\lambda}{c\,\mu}\right)^{K-c+1}\right]}{\left(1-\frac{\lambda}{c\,\mu}\right)^{2}}$$

sama dengan

$$\frac{\left[1-(K-c+1)\left(\frac{\lambda}{c\,\mu}\right)^{K-c}-\left[\frac{\lambda}{c\,\mu}-(K-c+1)\left(\frac{\lambda}{c\,\mu}\right)^{K-c+1}\right]\right]+\left[\frac{\lambda}{c\,\mu}-\left(\frac{\lambda}{c\,\mu}\right)^{K-c+1}\right]}{\left(1-\frac{\lambda}{c\,\mu}\right)^{2}}$$

sama dengan

$$\frac{\left[1-(K-c+1)\left(\frac{\lambda}{c\,\mu}\right)^{K-c}-\left[\frac{\lambda}{c\,\mu}-(K-c+1)\left(\frac{\lambda}{c\,\mu}\right)^{K-c+1}\right]\right]+\left[\frac{\lambda}{c\,\mu}-\left(\frac{\lambda}{c\,\mu}\right)^{K-c+1}\right]}{\left(1-\frac{\lambda}{c\,\mu}\right)^{2}}$$

sama dengan

$$\frac{1-(K-c+1)\left(\frac{\lambda}{c\,\mu}\right)^{K-c}+(K-c)\left(\frac{\lambda}{c\,\mu}\right)^{K-c+1}}{\left(1-\frac{\lambda}{c\,\mu}\right)^{2}}$$

Sehingga untuk $\frac{\lambda}{c\,\mu}\neq 1$

$$EN_q = p_0\ \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^{c}\frac{\lambda}{c\,\mu}\left[\frac{d}{d\left(\frac{\lambda}{c\,\mu}\right)}\left[\frac{\frac{\lambda}{c\,\mu}-\left(\frac{\lambda}{c\,\mu}\right)^{K-c+1}}{1-\frac{\lambda}{c\,\mu}}\right]\right]$$

$$EN_q = p_0\ \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^{c}\frac{\lambda}{c\,\mu}\ \frac{1-(K-c+1)\left(\frac{\lambda}{c\,\mu}\right)^{K-c}+(K-c)\left(\frac{\lambda}{c\,\mu}\right)^{K-c+1}}{\left(1-\frac{\lambda}{c\,\mu}\right)^{2}}$$

Untuk $\frac{\lambda}{c\ \mu} = 1$

$$EN_q = \frac{1}{c!} \left(\frac{\lambda}{\mu}\right)^c p_0 \sum_{j=1}^{K-c} \left[j \left(\frac{\lambda}{c\ \mu}\right)^j \right]$$

$$EN_q = p_0\ \frac{1}{c!} \left(\frac{\lambda}{\mu}\right)^c \sum_{j=1}^{K-c} \left[j\ (1)^j \right]$$

$$EN_q = p_0\ \frac{1}{c!} \left(\frac{\lambda}{\mu}\right)^c \sum_{j=1}^{K-c} j$$

$$EN_q = p_0\ \frac{1}{c!} \left(\frac{\lambda}{\mu}\right)^c \left[\frac{K-c}{2}\ (1 + K - c) \right]$$

Ekspektasi jumlah pelanggan antri ini fungsi dari (λ, μ, c, K) sehingga EN_q dapat dinotasikan secara lengkap sebagai:

$$
EN_q(\lambda, \mu, c, K) := \left|
\begin{array}{l}
\left(c \leftarrow \dfrac{c}{pelayan}\right) \wedge \left(K \leftarrow \dfrac{K}{pelanggan}\right) \\
\text{if } K \geq c \\
\quad \left|
\begin{array}{l}
\text{if } \dfrac{\lambda}{c\,\mu} \neq 1 \\
\quad \left|
\begin{array}{l}
p_0 \leftarrow \dfrac{1}{\left[\displaystyle\sum_{n=0}^{c-1} \left[\dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n\right]\right] + \dfrac{1}{c!}\left(\dfrac{\lambda}{\mu}\right)^c \dfrac{1-\left(\dfrac{\lambda}{c\,\mu}\right)^{K-c+1}}{1-\dfrac{\lambda}{c\,\mu}}} \\
S\rho \leftarrow \dfrac{\lambda}{c\,\mu}\left[\dfrac{1-(K-c+1)\left(\dfrac{\lambda}{c\,\mu}\right)^{K-c} + (K-c)\left(\dfrac{\lambda}{c\,\mu}\right)^{K-c+1}}{\left(1-\dfrac{\lambda}{c\,\mu}\right)^2}\right] \\
ENq \leftarrow p_0 \dfrac{1}{c!}\left(\dfrac{\lambda}{\mu}\right)^c S\rho
\end{array}
\right. \\
\text{if } \dfrac{\lambda}{c\,\mu} = 1 \\
\quad \left|
\begin{array}{l}
p_0 \leftarrow \dfrac{1}{\left[\displaystyle\sum_{n=0}^{c-1} \left[\dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n\right]\right] + \dfrac{1}{c!}\left(\dfrac{\lambda}{\mu}\right)^c (1+K-c)} \\
S\rho \leftarrow \left[\dfrac{K-c}{2}(1+K-c)\right] \\
ENq \leftarrow p_0 \dfrac{1}{c!}\left(\dfrac{\lambda}{\mu}\right)^c S\rho
\end{array}
\right. \\
ENq\ pelanggan
\end{array}
\right. \\
\text{"K harus lebih besar atau sama dengan c"} \quad otherwise
\end{array}
\right.
$$

Contoh 2

$\lambda = 14 \frac{pelanggan}{jam}$

laju datang pelanggan per satuan waktu.

λ menyatakan laju datang (*arrival rate*) yaitu jumlah pelanggan yang datang rata-rata per satuan waktu.

Laju datang rata-rata effektif

$$\lambda_{eff} = \sum_{n=0}^{K} (\lambda_n p_n)$$

$\mu = 4 \frac{pelanggan}{jam}$

laju layan pelanggan per satuan waktu.

μ menyatakan laju layan yaitu jumlah pelanggan yang dilayani rata-rata per satuan waktu.

Jumlah pelanggan maksimum yang diperbolehkan dalam sistem sebanyak K

$K := 7 pelanggan$ $\quad$ $\mathrm{K} := \frac{K}{pelanggan}$

$\lambda = 14 \frac{pelanggan}{jam}$ $\quad$ $\mu = 4 \frac{pelanggan}{jam}$

$c := 1 pelayan, 2 pelayan .. \mathrm{K}\ pelayan$

Jumlah pelayan $c =$ (*pelayan*)	Ekspektasi jumlah pelanggan antri $EN_q(\lambda, \mu, c, K) =$ (*pelanggan*)
1	5.6
2	3.807
3	1.9
4	0.718
5	0.219
6	0.046
7	0

Ekspektasi waktu antri

Ekspektasi waktu antri yaitu waktu rata-rata pelanggan berada dalam antrian.

$$ED = \frac{EN_q}{\lambda_{eff}}$$

di mana

$$\lambda_{eff} = \lambda\left(1 - p_K\right)$$

$$\lambda_{eff} = \lambda\left[1 - \frac{1}{c!\, c^{K-c}}\left(\frac{\lambda}{\mu}\right)^K p_0\right]$$

sehingga

$$ED = \frac{EN_q}{\lambda\left[1 - \frac{1}{c!\, c^{K-c}}\left(\frac{\lambda}{\mu}\right)^K p_0\right]}$$

Ekspektasi waktu antri ini fungsi dari (λ, μ, c, K) sehingga ED dapat dinotasikan secara lengkap sebagai:

$$ED(\lambda, \mu, c, K) = \frac{EN_q(\lambda, \mu, c, K)}{\lambda_{eff}}$$

$$ED(\lambda, \mu, c, K) := \left| \begin{array}{l} \left(c \leftarrow \dfrac{c}{pelayan}\right) \wedge \left(K \leftarrow \dfrac{K}{pelanggan}\right) \\ \left| \begin{array}{l} \left| \begin{array}{l} p_0 \leftarrow \dfrac{1}{\left[\displaystyle\sum_{n=0}^{c-1} \left[\dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n\right]\right] + \dfrac{1}{c!}\left(\dfrac{\lambda}{\mu}\right)^c \dfrac{1-\left(\dfrac{\lambda}{c\,\mu}\right)^{K-c+1}}{1-\dfrac{\lambda}{c\,\mu}}} \\ S\rho \leftarrow \dfrac{\lambda}{c\,\mu}\left[\dfrac{1-(K-c+1)\left(\dfrac{\lambda}{c\,\mu}\right)^{K-c} + (K-c)\left(\dfrac{\lambda}{c\,\mu}\right)^{K-c+1}}{\left(1-\dfrac{\lambda}{c\,\mu}\right)^2}\right] \\ ENq \leftarrow p_0 \dfrac{1}{c!}\left(\dfrac{\lambda}{\mu}\right)^c S\rho \\ ED \leftarrow \dfrac{ENq}{\lambda\left[1-\dfrac{1}{c^{K-c}\,c!}\left(\dfrac{\lambda}{\mu}\right)^K p_0\right]} \end{array} \right. \quad \text{if } \dfrac{\lambda}{c\,\mu} \neq 1 \\ \left| \begin{array}{l} p_0 \leftarrow \dfrac{1}{\left[\displaystyle\sum_{n=0}^{c-1} \left[\dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n\right]\right] + \dfrac{1}{c!}\left(\dfrac{\lambda}{\mu}\right)^c (1+K-c)} \\ S\rho \leftarrow \left[\dfrac{K-c}{2}(1+K-c)\right] \\ ENq \leftarrow p_0 \dfrac{1}{c!}\left(\dfrac{\lambda}{\mu}\right)^c S\rho \\ ED \leftarrow \dfrac{ENq}{\lambda\left[1-\dfrac{1}{c^{K-c}\,c!}\left(\dfrac{\lambda}{\mu}\right)^K p_0\right]} \end{array} \right. \quad \text{if } \dfrac{\lambda}{c\,\mu} = 1 \\ ED \end{array} \right. \quad \text{if } K \geq c \\ \text{"K harus lebih besar atau sama dengan c"} \quad \textit{otherwise} \end{array} \right.$$

Contoh 3

$\lambda = 14 \dfrac{pelanggan}{jam}$ laju datang pelanggan per satuan waktu.

λ menyatakan laju datang (*arrival rate*) yaitu jumlah pelanggan yang datang rata-rata per satuan waktu.

Laju datang rata-rata effektif $\lambda_{eff} = \sum_{n=0}^{K} (\lambda_n p_n)$

$\mu = 4 \dfrac{pelanggan}{jam}$ laju layan pelanggan per satuan waktu.

μ menyatakan laju layan yaitu jumlah pelanggan yang dilayani rata-rata per satuan waktu.

Jumlah pelayan

Ekspektasi waktu antri yaitu waktu rata-rata pelanggan berada dalam antrian

$c =$ (*pelayan*)	$ED(\lambda, \mu, c, K) =$ ($\frac{1}{pelanggan}$ *jam*)
1	1.4
2	0.482
3	0.174
4	0.058
5	0.017
6	$3.437 \cdot 10^{-3}$
7	0

Ekspektasi waktu sistem

Ekspektasi waktu sistem yaitu waktu rata-rata pelanggan berada dalam sistem.

$$EW(\lambda, \mu, c, K) = ED(\lambda, \mu, c, K) + \frac{1}{\mu}$$

di mana $ED(\lambda, \mu, c, K) = \dfrac{EN_q(\lambda, \mu, c, K)}{\lambda_{eff}}$

$$EW(\lambda ,\mu ,c,K) := \left| \begin{array}{l} \left(c \leftarrow \dfrac{c}{pelayan}\right) \wedge \left(K \leftarrow \dfrac{K}{pelanggan}\right) \\ \text{if } K \geq c \\ \quad \left| \begin{array}{l} \text{if } \dfrac{\lambda}{c\ \mu} \neq 1 \\ \quad \left| \begin{array}{l} p_0 \leftarrow \dfrac{1}{\left[\displaystyle\sum_{n=0}^{c-1} \left[\dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n\right]\right] + \dfrac{1}{c!}\left(\dfrac{\lambda}{\mu}\right)^c \dfrac{1-\left(\dfrac{\lambda}{c\ \mu}\right)^{K-c+1}}{1-\dfrac{\lambda}{c\ \mu}}} \\ S\rho \leftarrow \dfrac{\lambda}{c\ \mu}\left[\dfrac{1-(K-c+1)\left(\dfrac{\lambda}{c\ \mu}\right)^{K-c} + (K-c)\left(\dfrac{\lambda}{c\ \mu}\right)^{K-c+1}}{\left(1-\dfrac{\lambda}{c\ \mu}\right)^2}\right] \\ ENq \leftarrow p_0 \dfrac{1}{c!}\left(\dfrac{\lambda}{\mu}\right)^c S\rho \\ EW \leftarrow \dfrac{ENq}{\lambda\left[1-\dfrac{1}{c^{K-c}\ c!}\left(\dfrac{\lambda}{\mu}\right)^K p_0\right]} + \dfrac{1}{\mu} \end{array} \right. \\ \text{if } \dfrac{\lambda}{c\ \mu} = 1 \\ \quad \left| \begin{array}{l} p_0 \leftarrow \dfrac{1}{\left[\displaystyle\sum_{n=0}^{c-1} \left[\dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n\right]\right] + \dfrac{1}{c!}\left(\dfrac{\lambda}{\mu}\right)^c (1+K-c)} \\ S\rho \leftarrow \left[\dfrac{K-c}{2}(1+K-c)\right] \\ ENq \leftarrow p_0 \dfrac{1}{c!}\left(\dfrac{\lambda}{\mu}\right)^c S\rho \\ EW \leftarrow \dfrac{ENq}{\lambda\left[1-\dfrac{1}{c^{K-c}\ c!}\left(\dfrac{\lambda}{\mu}\right)^K p_0\right]} + \dfrac{1}{\mu} \end{array} \right. \\ EW \end{array} \right. \\ \text{"K harus lebih besar atau sama dengan c"} \quad \textit{otherwise} \end{array} \right.$$

Contoh 4

$\lambda = 14 \dfrac{pelanggan}{jam}$ laju datang pelanggan per satuan waktu.

λ menyatakan laju datang (*arrival rate*) yaitu jumlah pelanggan yang datang rata-rata per satuan waktu.

Laju datang rata-rata effektif $\lambda_{eff} = \sum_{n=0}^{K} (\lambda_n p_n)$

Dalam hal ini $\lambda_n = \lambda$ konstan untuk $n \geq 0$.

$\mu = 4 \dfrac{pelanggan}{jam}$ laju layan pelanggan per satuan waktu.

μ menyatakan laju layan yaitu jumlah pelanggan yang dilayani rata-rata per satuan waktu.

Jumlah pelayan

Ekspektasi waktu sistem yaitu waktu rata-rata pelanggan berada dalam sistem

$c =$ (*pelayan*)	$EW(\lambda, \mu, c, K) =$ ($\dfrac{1}{pelanggan}$ *jam*)
1	1.65
2	0.732
3	0.424
4	0.308
5	0.267
6	0.253
7	0.25

Ekspektasi jumlah pelanggan sistem

Ekspektasi jumlah pelanggan sistem besarnya

$$EN(\lambda, \mu, c, K) = \lambda_{eff}\ EW(\lambda, \mu, c, K)$$

$$
\begin{aligned}
EN(\lambda,\mu,c,K) :=\ & \left(c \leftarrow \frac{c}{pelayan}\right) \wedge \left(K \leftarrow \frac{K}{pelanggan}\right)\\
& \text{if } K \ge c\\
& \quad \text{if } \frac{\lambda}{c\,\mu} \ne 1\\
& \quad\quad p_0 \leftarrow \frac{1}{\left[\sum_{n=0}^{c-1}\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]\right] + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{1-\left(\frac{\lambda}{c\,\mu}\right)^{K-c+1}}{1-\frac{\lambda}{c\,\mu}}}\\
& \quad\quad S\rho \leftarrow \frac{\lambda}{c\,\mu}\left[\frac{1-(K-c+1)\left(\frac{\lambda}{c\,\mu}\right)^{K-c} + (K-c)\left(\frac{\lambda}{c\,\mu}\right)^{K-c+1}}{\left(1-\frac{\lambda}{c\,\mu}\right)^2}\right]\\
& \quad\quad ENq \leftarrow p_0\,\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c S\rho\\
& \quad\quad EN \leftarrow \lambda\left[1-\frac{1}{c^{K-c}\,c!}\left(\frac{\lambda}{\mu}\right)^K p_0\right]\left[\frac{ENq}{\lambda\left[1-\frac{1}{c^{K-c}\,c!}\left(\frac{\lambda}{\mu}\right)^K p_0\right]} + \frac{1}{\mu}\right]\\
& \quad \text{if } \frac{\lambda}{c\,\mu} = 1\\
& \quad\quad p_0 \leftarrow \frac{1}{\left[\sum_{n=0}^{c-1}\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]\right] + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c (1+K-c)}\\
& \quad\quad S\rho \leftarrow \left[\frac{K-c}{2}\,(1+K-c)\right]\\
& \quad\quad ENq \leftarrow p_0\,\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c S\rho\\
& \quad\quad EN \leftarrow \lambda\left[1-\frac{1}{c^{K-c}\,c!}\left(\frac{\lambda}{\mu}\right)^K p_0\right]\left[\frac{ENq}{\lambda\left[1-\frac{1}{c^{K-c}\,c!}\left(\frac{\lambda}{\mu}\right)^K p_0\right]} + \frac{1}{\mu}\right]\\
& \quad EN\ pelanggan\\
& \text{"K harus lebih besar atau sama dengan c"}\ \ \textit{otherwise}
\end{aligned}
$$

Contoh 5

$\lambda = 14 \frac{pelanggan}{jam}$

laju datang pelanggan per satuan waktu.

λ menyatakan laju datang (*arrival rate*) yaitu jumlah pelanggan yang datang rata-rata per satuan waktu.

Laju datang rata-rata effektif

$$\lambda_{eff} = \sum_{n=0}^{K} \left(\lambda_n p_n\right)$$

$\mu = 4 \frac{pelanggan}{jam}$

laju layan pelanggan per satuan waktu.

μ menyatakan laju layan yaitu jumlah pelanggan yang dilayani rata-rata per satuan waktu.

Jumlah pelayan

$c =$ *pelayan*

Ekspektasi jumlah pelanggan sistem

$EN(\lambda, \mu, c, K) =$ *pelanggan*

c (pelayan)	$EN(\lambda, \mu, c, K)$ (pelanggan)
1	6.6
2	5.784
3	4.633
4	3.837
5	3.495
6	3.385
7	3.361