## PROBABILITAS DAN STATISTIKA

# DISTRIBUSI NORMAL BAKU

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**Fungsi padat probabilitas $Z$ yang berdistribusi normal baku**

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} \qquad -\infty < z < \infty$$

Bila ditulis

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} dz = I$$

*I* dikalikan *I*

$$I\ I = \left( \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} dz \right) \left( \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y^2} dy \right)$$

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} \right) \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y^2} \right) dz\, dy$$

$$I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(z^2+y^2\right)} dz\, dy$$

Jika diubah dari *curvilinear coordinates* ke *polar coordinates* (baca Kaplan, *Advanced Calculus*, second edition, 1981, p. 270-272)

$z = r\cos(\theta)$ dan $y = r\sin(\theta)$ maka

$$\int_{R_{zy}}^{\blacksquare}\int_{R_{zy}}^{\blacksquare} f\,(z\,,y)\,dz\,dy = \int_{R_{r\theta}}^{\blacksquare}\int_{R_{r\theta y}}^{\blacksquare} f\,(r\,\cos(\theta)\,,r\,\sin(\theta))\left\|\begin{bmatrix}\frac{d}{dr}(r\,\cos(\theta)) & \frac{d}{d\theta}(r\,\cos(\theta))\\ \frac{d}{dr}(r\,\sin(\theta)) & \frac{d}{d\theta}(r\,\sin(\theta))\end{bmatrix}\right\|\,dr\,d\theta$$

$$\int_{R_{zy}}^{\blacksquare}\int_{R_{zy}}^{\blacksquare} f\,(z\,,y)\,dz\,dy = \int_{R_{r\theta}}^{\blacksquare}\int_{R_{r\theta y}}^{\blacksquare} f\,(r\,\cos(\theta)\,,r\,\sin(\theta))\left|\begin{pmatrix}\cos(\theta) & -r\,\sin(\theta)\\ \sin(\theta) & r\,\cos(\theta)\end{pmatrix}\right|\,dr\,d\theta$$

determinan

$$det\left[\left|\begin{pmatrix}\cos(\theta) & -r\,\sin(\theta)\\ \sin(\theta) & r\,\cos(\theta)\end{pmatrix}\right|\right] = \left[r\,\cos(\theta)^2-\left(-r\,\sin(\theta)^2\right)\right] = r$$

sehingga

$$\int_{R_{zy}}^{\blacksquare}\int_{R_{zy}}^{\blacksquare} f\,(z\,,y)\,dz\,dy = \int_{R_{r\theta}}^{\blacksquare}\int_{R_{r\theta y}}^{\blacksquare} f\,(r\,\cos(\theta)\,,r\,\sin(\theta))\,r\,dr\,d\theta$$

$-\infty < z < \infty$ $\qquad$ $-\infty < y < \infty$

menjadi

$0 \le \theta \le 2\,\pi$ $\qquad$ $0 \le r \le \infty$

maka

$$I^2 = \frac{1}{2\,\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(z^2+y^2\right)}\,dz\,dy = \frac{1}{2\,\pi}\int_{0}^{2\,\pi}\int_{0}^{\infty} e^{-\frac{1}{2}\left[(r\,\cos(\theta))^2+(r\,\sin(\theta))^2\right]}\,r\,dr\,d\theta$$

$$I^2 = \frac{1}{2\,\pi}\int_{0}^{2\,\pi}\int_{0}^{\infty} e^{-\frac{1}{2}r^2}\,r\,dr\,d\theta$$

sedangkan

$$\int_{0}^{\infty} e^{-\frac{1}{2}r^2}\,r\,dr = -\int_{0}^{\infty} e^{-\frac{1}{2}r^2}\,r\,d\left(-\frac{1}{2}r^2\right) = -\left(e^{-\frac{1}{2}\infty^2}-e^{-\frac{1}{2}0^2}\right) = -(0-1)$$

sehingga

$$I^2 = \frac{1}{2\pi}\int_0^{2\pi} (1)\, d\theta = \frac{1}{2\pi}(2\pi - 0) = 1$$

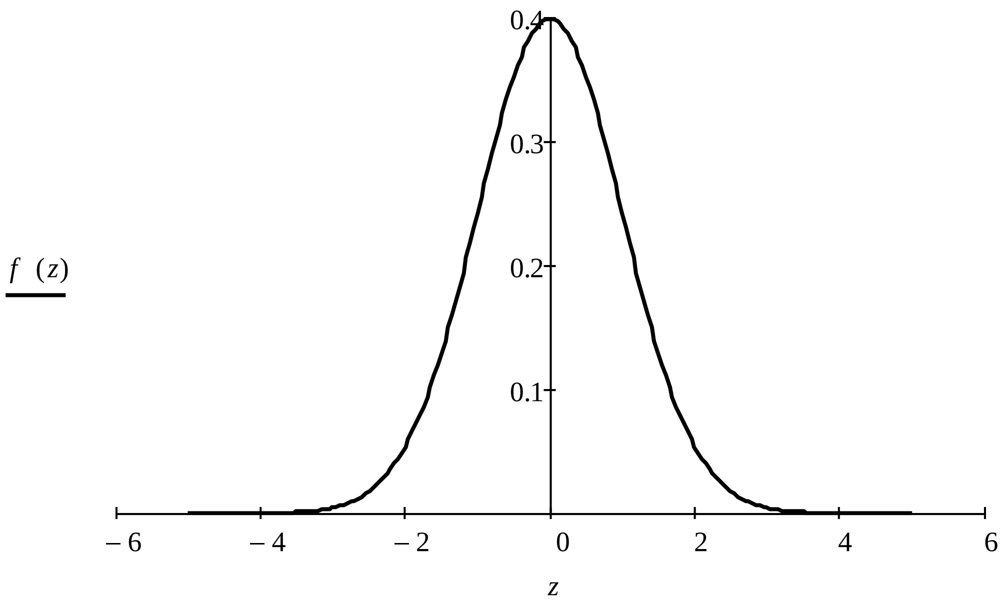
$$I = 1$$

Variabel acak *Z* yang berdistribusi normal baku mempunyai fungsi padat probabilitas

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \qquad -\infty < z < \infty$$

$$f(z) := \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$ $E(Z) = 0$ Mean *Z* atau ekspektasi *Z*

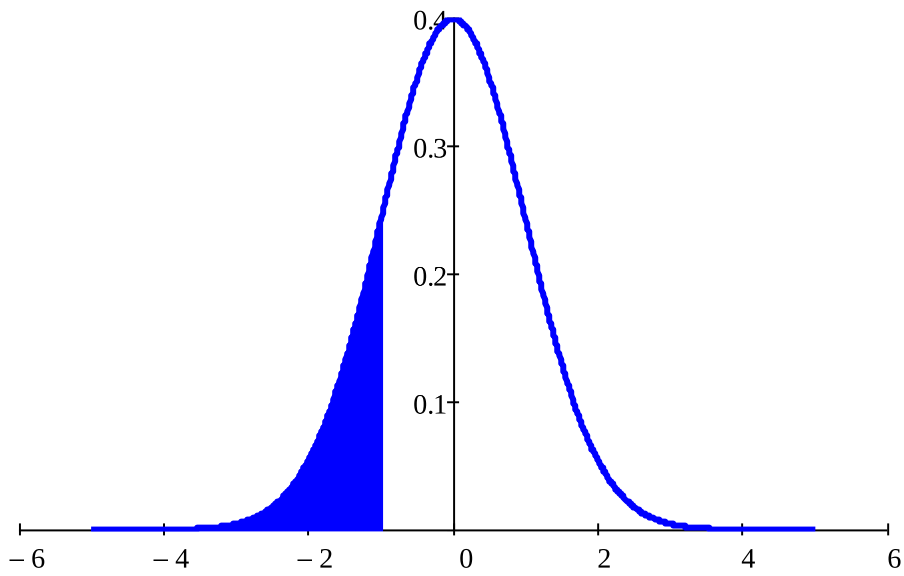
$f(0) = 0.39894$ $V(Z) = 1$ Variansi *Z*




$a := -5$ $b := 5$ $c := -5$ $d := -1$

$z1 := a, a + 0.01 .. b$ $z2 := c, c + 0.01 .. d$

$$f(z1) := \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z1^2} \qquad f(z2) := \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z2^2}$$




$$\int_{c}^{d} f(z)\,dz = 0.15865$$

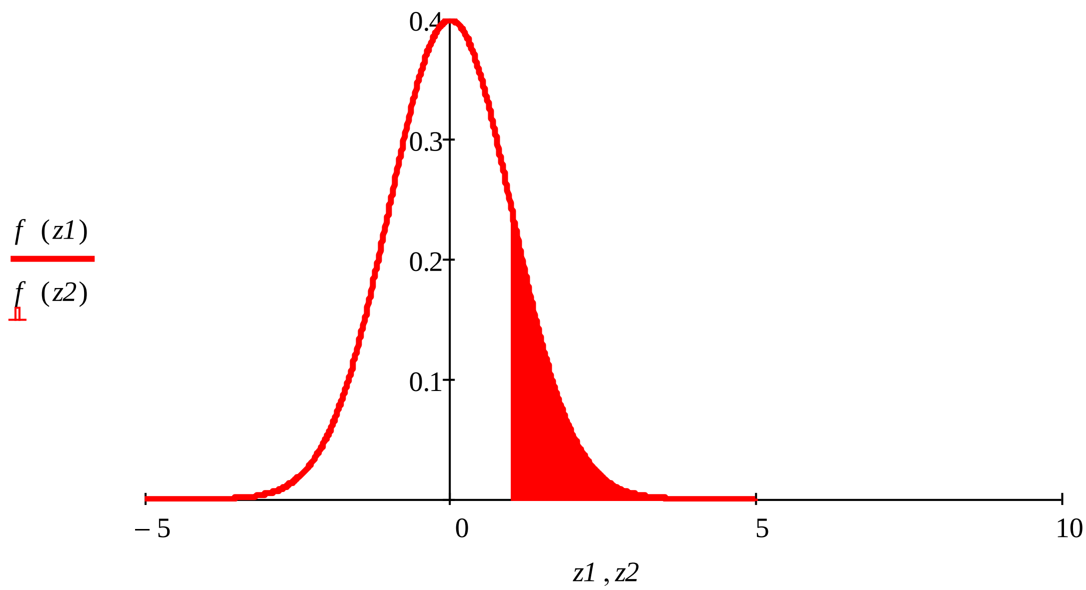
$c = -5$

$d = -1$

$a := -5 \qquad b := 5 \qquad\qquad c := 1 \qquad d := 5$

$z1 := a, a + 0.01 .. b \qquad\qquad z2 := c, c + 0.01 .. d$

$$f(z1) := \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2} z1^2} \qquad\qquad f(z2) := \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2} z2^2}$$



$$\int_c^d f(z)\,dz = 0.15865$$

$c = 1 \qquad d = 5$

$$IN(a) := \int_0^a \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,dz \text{ simplify } \rightarrow \frac{erf\left(\frac{\sqrt{2}\,a}{2}\right)}{2}$$

$$I(z) := \int_0^\infty e^{-\frac{1}{2}z^2}\,dz \text{ simplify } \rightarrow \frac{\sqrt{2}\sqrt{\pi}\ erf\left(50\sqrt{2}\right)}{2}$$

$$erf\left(500\sqrt{2}\right) = 1$$

$$Integ(\infty) := \int_{-\infty}^\infty e^{-\frac{1}{2}z^2}\,dz \text{ simplify } \rightarrow \sqrt{2}\sqrt{\pi}$$

$Integ(\infty) = 2.50663 \qquad \sqrt{2\pi} = 2.50663$

$$I(\infty) := \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,dz \text{ simplify } \rightarrow 1$$

$I(\infty) = 1$

*Error function*

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \qquad erf(\infty) = 1$$

*Error function complementary*

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt = 1 - erf(x)$$

$$Int(\infty) := \int_0^\infty e^{\frac{-r^2}{2}}\,r\,dr \text{ simplify } \rightarrow 1 \qquad Int(\infty) = 1$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = 1$$

$e = 2.71828$

$\pi = 3.14159$

$Z := str2vec(\text{"Z"})_{ORIGIN}$

$mean(Z)$ atau $E(Z)$

$$E(VariabelAcak) := \begin{cases} \int_{-\infty}^{\infty} z\ f(z)\, dz & \text{if } VariabelAcak = str2vec(\text{"Z"})_{ORIGIN} \\ error(\text{"Tidak didifinisikan"}) & \text{otherwise} \end{cases}$$

$E(Z) = 0$

$$\int_{-\infty}^{\infty} z\ f(z)\, dz = 0$$

**Variansi *Z*** atau $V(Z)$

$$V(VariabelAcak) := \begin{cases} \int_{-\infty}^{\infty} z^2 f(z)\, dz \ldots + -\left(\int_{-\infty}^{\infty} z\ f(z)\, dz\right)^2 & \text{if } VariabelAcak = str2vec(\text{"Z"})_{ORIGIN} \\ error(\text{"Tidak didifinisikan"}) & \text{otherwise} \end{cases}$$

$V(Z) = 1$

**Momen ke-*r*:**

$$momen(r) := \int_{-\infty}^{\infty} z^r \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\right) dz$$

$momen(1) = 0$

$momen(2) = 1$

$momen(3) = 0$

$momen(4) = 3$

$momen(5) = 0$

$momen(6) = 15$

$momen(7) = 0$

$momen(8) = 105$

**Fungsi generator momen** $M_Z(t)$ atau $E\left(e^{t\,Z}\right)$

$$M_Z(t) = E\left(e^{t\,Z}\right) = \int_{-\infty}^{\infty} e^{t\,z}\, f\,(z)\,dz$$

$$M_Z(t) = \int_{-\infty}^{\infty} e^{t\,z}\left(\frac{1}{\sqrt{2\,\pi}}\; e^{-\frac{1}{2}\,z^2}\right)dz$$

$$M_Z(t) = e^{\frac{1}{2}\,t^2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\,\pi}}\; e^{-\frac{1}{2}\,(z-t)^2}\,dz$$

jika ditulis

$z - t = w$ maka $dz = dw$

dan

$$M_Z(t) = e^{\frac{1}{2}\,t^2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\,\pi}}\; e^{-\frac{1}{2}\,w^2}\,dw$$

$$M_Z(t) = e^{\frac{1}{2}\,t^2}\;(1)$$

$$M_Z(t) = e^{\frac{1}{2}\,t^2}$$